M.Math. Algebra II - Midsemestral Exam
February 26, 2020
Instructor — B. Sury
Maximum Marks 120 - Be BRIEF!
You may quote and use without proof results which are not almost equivalent to the given problem.

Q 1.
(8 + 12 marks)
(a) If L/K is a finite extension and A is an intermediate subring (that is, $K \subseteq A \subseteq L$), prove that A must be a field.
(b) If f is an irreducible polynomial of degree n over a field K, then prove that its splitting field over K has degree dividing $n!$.

OR

(10+10 marks)
(a) Show that the splitting of the following polynomial $x^4 + x^2 + 1$ over $\mathbb{Q}$ is generated by a primitive cube root of unity.
(b) Determine the cardinality of the splitting field of $x^3 - 2$ over $\mathbb{F}_7$.

.

Q 2.
(19 marks)
If K is an extension of degree 6 over $\mathbb{Q}$, prove that the polynomial $f(X) = X^5 - 2$ must be irreducible in $K[X]$.

OR

(21 marks)
Let $a \in K$ where char $K = p$. Let $f = X^p - X - a \in K[X]$. Show that each field extension L of K has the property that either f has no roots in L or it splits into linear factors over L.

Q 3.
(19 marks)
Suppose L/K is an algebraic extension and let $a \in L$. If a is purely inseparable (that is, min (a, K) has only one root), prove that either $a \in K$ or K must have prime characteristic p and $a^{p^n} \in K$ for some n.

OR

(21 marks)
Prove that if L/K is an algebraic extension, and S is the separable closure of L over K, then L is purely inseparable over S.

Q 4.
(20 marks)
Let L/K be a separable extension of degree n. Suppose $\sigma_1, \cdots, \sigma_n$ are the K-algebra homomorphisms from L into an algebraic closure $\bar{K}$ containing L. For any n-tuple $(a_1, \cdots, a_n) \in L^n$, show that the matrix whose (i,j)-th entry is $tr_{L/K}(a_i a_j)$ has determinant equal to det A^2 where A is the matrix with (i,j)-th entry $\sigma_i(a_j)$.

OR

(18 marks)
Let $K = \mathbb{Q}(e^{2i\pi/p})$ where p is an odd prime. Let $E = K \cap \mathbb{R}$. Determine $N_{K/E}(e^{2i\pi/p})$. Further, find an element t of K with $N_{K/\mathbb{Q}}(t) = p$.

Q 5.
(20 marks)
If $f \in \mathbb{Q}[X]$ has splitting field of odd degree over $\mathbb{Q}$, prove that all the roots of f must be real.

OR

(20 marks)
If K is the splitting field of an irreducible polynomial $f \in \mathbb{Q}[X]$ such that all subgroups of $Gal(K/\mathbb{Q})$ are normal, prove that $K = \mathbb{Q}(\alpha)$ for ANY root of f in K.

Q 6.
(3+3+4+5+6)
Give examples (no need to prove) of field extensions L/K with:
(i) L/K normal but not Galois.
(ii) L/K separable but not Galois.
(iii) $[L : K]$ finite, but with infinitely many intermediate extensions.
(iv) L/K infinite, but with L/E finite for every intermediate extension $E \neq K$.
(v) An infinite algebraic extension of $\mathbb{F}_p$ which is not algebraically closed.

OR

(20 marks)
Let $f_1, f_2 \in K[X]$, where K is an arbitrary field. Let α be a root of f_1 in a fixed algebraic closure of K. Then, the composition $f_1 \circ f_2$ is irreducible over K if, and only if, f_1 is irreducible over K and $f_2 - \alpha$ is irreducible over $K(\alpha)$.

OR

(20 marks)
Show that for any $n \geq 1$, the field $\mathbb{C}(X)$ is Galois over $\mathbb{C}(X^n + X^{-n})$ with Galois group D_n.